Comment on the quantum nature of angular momentum using a coupled-boson representation

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Abstract

A simple approach for understanding the quantum nature of angular momentum and its reduction to the classical limit is presented based on Schwinger's coupled-boson representation. This approach leads to a straightforward explanation of why the square of the angular momentum in quantum mechanics is given by $j(j+1)$ instead of just j^2, where j is the angular momentum quantum number.

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Angular momentum in quantum mechanics plays important roles in the treatment of central force motion, molecular motion, spin dynamics, and the quantum dynamics of coupled multi-level quantum systems. Angular momentum oscillators have been a prime basis for representing atomic species in quantum electronics and quantum optics. Yet, the "intuitive" understanding of the quantum-mechanical angular momentum and its reduction to the classical limit is still not often discussed in introductory textbooks on quantum mechanics. In this paper, we will provide a simple approach for understanding the quantum nature of angular momentum and its reduction to the classical limit by considering the coupled-boson representation of angular momentum, noted by Schwinger in 1960s [1].

It is well-known that the square of angular momentum has eigenvalue equations,

$$\hat{J}^2 \left|jm_j\right\rangle = j(j+1)\hbar^2 \left|jm_j\right\rangle \tag{1}$$

and

$$\hat{J}_z \left|jm_j\right\rangle = m_j\hbar \left|jm_j\right\rangle \; ; \tag{2}$$

with $m_j = -j, -j+1, \cdots, j$; here $|jm_j\rangle$ is the common set of eigenstates of $\hat{J}^2$ and $\hat{J}_z$ since $[\hat{J}^2, \hat{J}_z] = 0$. In introductory textbooks, $\langle\hat{J}^2\rangle = j(j+1)$, instead of j^2, is noted as a genuine quantum-mechanical result without any easy way of understanding why. In [2], Milonni provided a very simple and fancy "derivation" of why $j(j+1)$ is the quantum result; it followed from the symmetry $\langle\hat{J}^2\rangle = 3\,\langle\hat{J}_z^2\rangle$ and assuming the allowance of the only $2j+1$ values of $\hat{J}_z$ with a well-known sum rule,

$$\sum_{m_j=-j}^{j} m_j^2 \;=\; \frac{1}{3}\,j(j+1)(2j+1)\,. \tag{3}$$

But he could not show, as he stated, why the $2j+1$ values only are allowed.

Schwinger noted that the quantum-mechanical angular momentum can be obtained by using creation and annihilation operators of two 1-dimensional harmonic oscillators in the form of $\hat{J}_\mu \propto \hat{a}_1^\dagger \otimes \hat{a}_2$, where $\hat{a}_1$, $\hat{a}_2$ are the annihilation operators of two linear harmonic oscillators, respectively. This is explicitly given as follows: [1]

$$\begin{aligned} \hat{J}_x &= \frac{\hbar}{2}\left(\hat{a}_1^\dagger \otimes \hat{a}_2 + \hat{a}_1 \otimes \hat{a}_2^\dagger\right) \quad , \quad \hat{J}_z = \frac{\hbar}{2}\left(\hat{n}_1 - \hat{n}_2\right) \\ \hat{J}_y &= \frac{\hbar}{2i}\left(\hat{a}_1^\dagger \otimes \hat{a}_2 - \hat{a}_1 \otimes \hat{a}_2^\dagger\right) \quad , \quad \hat{J} = \frac{\hbar}{2}\left(\hat{n}_1 + \hat{n}_2\right)\,, \end{aligned} \tag{4}$$

where $\hat{n}_k = \hat{a}_k^\dagger\,\hat{a}_k$ with $k = 1,2$ is an occupation number operator of each 1-dimensional oscillator k. By using $\left[\hat{a}_k\,, \hat{a}_l^\dagger\right] = \delta_{kl}$ the operators $\hat{J}_\mu$ satisfy the usual angular-momentum commutation relations. The total occupation number, $n = n_1 + n_2$ with $n_k = 0, 1, 2\cdots$ as well as the operator $\hat{J}$ in (4) is a constant of motion for Hamiltonian having no interactions with external fields (energy conservation).

From this we clearly see a remarkable correspondence between two linear oscillators and an angular-momentum oscillator. Such a correspondence was utilized early-on by Holstein and Primakoff in connection with the theory of spin waves and magnon dynamics in ferromagnetic systems [3]; this correspondence was also well established in the theory of SU symmetries [4]. However, Schwinger's discourse on this correspondence shows the seamless general connection between angular momentum and two independent harmonic oscillators in a simple and direct fashion [5]. Therefore, the Schwinger approach is highlighted in this comment for pedagogical value.

In considering the commutation relation $\left[\hat{a}_k, \hat{a}_l^\dagger\right] = \epsilon\,\delta_{kl}$ with $\epsilon = 1, 0$, respectively, it is observed that the quantum as well as the classical behavior are depicted. By using (4) with this commutation relation, we find, after a minor calculation, that

$$\hat{J}^2 \;=\; \hat{J}_x^2 \,+\, \hat{J}_y^2 \,+\, \hat{J}_z^2 \;=\; \hat{J}\left(\hat{J} \,+\, \epsilon\,\hat{1}\right)\,. \tag{5}$$

Here, we clearly see that the square of the quantum-mechanical angular momentum must be given by $j(j+1)$ when $\epsilon = 1$, with j^2 expressing the classical angular momentum when $\epsilon = 0$; here j is one-half of the total oscillator occupation number, $j = \frac{1}{2}(n_1 + n_2)$. From this property of j, it also turns out that the quantum-mechanical angular momentum is evidently specified by one of the only allowed values $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$ for all possible values of $n_1, n_2 = 0, 1, 2, \cdots$. Thus, the allowed quantum numbers of the angular momentum are integer (e.g. for orbital angular momenta) or half-integer values as naturally concluded from the coupled-boson representation. Furthermore, since for $\epsilon = 0$, the operators $\hat{J}_\mu$ commute with each other, and the operators $\hat{n}_j$ in $\hat{J}_z$ and $\hat{J}$ lose their meaning as number operators, it is naturally expected that for the classical case j assumes continuous values.

In the quantum case, for a fixed value of n, and therefore $j = \frac{n}{2}$, the $2j+1$ different number states are available in the sublets of constant j: $(0, 2j), (1, 2j-1), \cdots, (2j, 0)$, which can be characterized by $m_j = -j, -j+1, \cdots, j$ of the $\hat{J}_z$ in (4), respectively. Therefore the total system with the conservation of the angular momentum $\hat{J}$ in (4) is naturally given by the $(2j+1)$-levels. This picture clearly provides the answer of why those values of m_j are only allowed for $\hat{J}_z$, which was missing in [2]. By using the sum rule (3) as in [2], we arrive at $\langle \hat{J}^2 \rangle = j(j+1)\hbar^2$, thus obtaining the desired value $j(j+1)$ in a simple manner.

Based on the above result, it is also interesting to consider the angle between $\hat{J}_z$ and $\hat{J}$ given by

$$\cos\vartheta_{m_j} \equiv \frac{J_z}{|\hat{J}|} = \frac{m_j\,\hbar}{\sqrt{j\,(j+\epsilon)\,\hbar^2}} = \frac{m_j/j}{\sqrt{1+\epsilon/j}}\,, \tag{6}$$

independent of $\hbar$. As a maximum and a minimum value of $\cos\vartheta_{m_j}$ we have $\cos\vartheta_j = \frac{1}{\sqrt{1+\epsilon/j}}$ and $\cos\vartheta_{-j} = \frac{-1}{\sqrt{1+\epsilon/j}}$, respectively. In the quantum regime, $\cos\vartheta_{\pm j}$ can never reach the classically achievable extrema, $\vartheta_j = 0$, $\vartheta_{-j} = \pi$, because $\epsilon/j \neq 0$ in observance of the uncertainty relation; in the classical limit, $\epsilon/j = 0$, and the classical extrema are then realizable. Since $\cos\vartheta_{\pm j}$ is a universal function of ϵ/j, independent of $\hbar$, the classical limit can be achieved either for $\epsilon = 0$, $j =$ finite or for $\epsilon = 1$, $j \to \infty$.

In this comment we have simply shown, on the basis of the coupled-boson representation of angular momentum, why only $2j+1$ values of $\hat{J}_z$ are available in the quantum-mechanical angular momentum with $\langle \hat{J}^2 \rangle = j(j+1)\hbar^2$; also the quantum nature of angular momentum and its reduction to the classical limit was easily demonstrated.

References

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[4] See, e.g., F. M. Fernández and E. A. Castro, *Algebraic Methods in Quantum Chemistry and Physics*, (CRC Press, New York, 1996), pp 47.

[5] See Ref. [1], page 155 on the subject of “Decomposition into spin”; use of the term “Doesn’t this set bells ringing?” notes the *pedagogical* tone set by Schwinger’s analysis.